Special Relativity Exercises

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1 Introduction

1.1 Introduction

Dear reader, this collection of exercises is dedicated exclusively to special relativity, according to the level that can be tackled at the scientific high school. The collection is taken from “Resolved exercises in physics”: I extracted it from that to make it easier to find and use on the internet. All the considerations made there also apply to these exercises.

I hope that what is reported in this work is if not helpful at least not harmful. To improve what is written and highlight any errors, do not hesitate to write to me.

I apologize for my bad English.

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1.2 Notations and precision in calculations

Throughout this work the SI has been followed, using the siunitx package in Xetex for its drafting.

As regards the precision of the calculations reported, it was decided to indicate the intermediate steps with more precision than the usual rules for the propagation of errors would indicate. The final results are instead reported, preferably in scientific notation, with a number of significant digits never lower than the precision of the starting data.

1.3 License and Copyright

2 Relativity

2.1 Time dilation, length contraction

Esercizio 1 *Andrea is an astronomer and Marco an astronaut: they have both* 40 *years. Marco is recruited to make a journey to the distant star Sirius* 8.6 *light-years, with a spaceship capable of going at the speed of* 0.95 c*. Finds:*

1. *how long the journey takes for Andrea and how long for Marco;*
2. *how much is the distance covered for Andrea and how much for Marco;*
3. *the age of both when the star is reached.*

Suppose that both Andrea and Marco are each in an inertial frame of reference. Time and distance traveled are measured between two events: departure from Earth and arrival on Sirius.

1. For Andrea, time is measured with a clock placed in front of him. However, the event of departure and arrival take place in different places: it is not a question of a proper time. The elapsed time can be derived from the speed definition.

$$v = \frac{\Delta x_0}{\Delta t} \tag{2.1}$$

where v is the speed of the spacecraft, Δx_0 the distance between the Earth and Sirius, measured while standing still on the Earth, is Δt the elapsed time.

$$\Delta t = \frac{\Delta x_0}{v} = \frac{8.6\ \text{al}}{0.95\ \text{c}} = 9.1\ \text{years} = 2.9 \times 10^8\ \text{s} \tag{2.2}$$

If Andrea, on Earth, looks at the clock that Marco, on the spaceship, is carrying with him, he will see that the elapsed time (in this case proper time because it measures the time interval between two events that take place in the same place) is:

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \Delta t\sqrt{1-\frac{v^2}{c^2}} = 2.9 \times 10^8\ \text{s}\sqrt{1-\left(\frac{0.95\ \text{c}}{\text{c}}\right)^2} = 8.9 \times 10^7\ \text{s} \tag{2.3}$$

This is the elapsed time for Marco and it is a proper time because the event of departure and arrival take place, in his reference system, in the same position.

2. Whoever measures the distance between the Earth and Sirius from the Earth's reference system sees both the place of arrival and departure stationary in their own reference system: this distance is a proper length. It is worth:

$$l_0 = 8.6\ \text{ly} = 8.1 \times 10^{16}\ \text{m} \tag{2.4}$$

Marco on the other hand, in his reference system, is as if he were standing still and saw both the starting point and the arrival point moving at the speed v= 0.95 c: the distance measured by him is not a proper length. The distance he traveled can be obtained from the definition of speed, considering the time Δt_0 measured by him to arrive at destination.

$$v = \frac{\Delta x}{\Delta t_0} \tag{2.5}$$

$$l = \Delta x = v\Delta t = 0.95\,\mathrm{c} \cdot 8.9 \times 10^7\,\mathrm{s} = 2.5 \times 10^{16}\,\mathrm{m} \tag{2.6}$$

We observe that the relative velocity between Marco's and Andrea's reference systems can only be, in modulus, the same for both. We can derive this distance also considering the phenomenon of length contraction.

$$l = \frac{l_0}{\gamma} = l_0\sqrt{1 - \frac{v^2}{\mathrm{c}^2}} = 8.1 \times 10^{16}\,\mathrm{m}\sqrt{1 - \left(\frac{0.95\,\mathrm{c}}{\mathrm{c}}\right)^2} = 2.5 \times 10^{16}\,\mathrm{m} \tag{2.7}$$

3. Finally, when the spaceship reaches its destination, Andrea's age will be:

$$t = 40\,\text{years} + \Delta t = 49\,\text{years} \tag{2.8}$$

Instead the age of Marco:

$$t' = 40\,\text{years} + \Delta t_0 = 43\,\text{years} \tag{2.9}$$

Esercizio 2 *A spacecraft is sent from Earth to Jupiter; the spacecraft moves with a constant speed $v = 25\,\mathrm{km/s}$. Determine the time difference between what is indicated by a clock on Earth, after a year of travel, and that placed on the spacecraft.*

We have two events: the position of the spacecraft at departure and the one after one year. The time Δt_0 spent on the spacecraft (still unknown) is a proper time: the two events take place, for this reference system, in the same position. The time Δt spent on Earth is instead a not proper time.

Therefore, the following relationship exists between the two times:

$$\Delta t = \gamma \Delta t_0 \tag{2.10}$$

Let's calculate the factor γ.

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1 - \left(\frac{25\,\mathrm{km/s}}{2.99 \times 10^5\,\mathrm{km/s}}\right)^2}} = 1.0000000035 \tag{2.11}$$

However, there is a problem with this result: an ordinary scientific calculator will give only one as a result, as it cannot handle enough significant figures to show the final figures of the factor calculated here.

To solve this problem, and when speeds are much lower than the speed of light, we can use a series expansion. It can be shown that, if $x \ll 1$, the following relation holds with good approximation:

$$(1+x)^{\alpha} \simeq 1 + \alpha x \tag{2.12}$$

If we apply this relation to the gamma factor we can write:

$$\frac{1}{\sqrt{1-\beta^2}} = \left(1-\beta^2\right)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}\beta^2 \tag{2.13}$$

$$\gamma \simeq 1 + \frac{1}{2}\left(\frac{25\,\mathrm{km/s}}{2.99 \times 10^5\,\mathrm{km/s}}\right)^2 = 1 + 0.0000000035 \tag{2.14}$$

So if a year are 31536000 s, the time lag Δt_x between the two clocks is:

$$\Delta t_x = \Delta t - \Delta t_0 = \gamma \Delta t_0 - \Delta t_0 = \Delta t_0(\gamma - 1) = 31536000\,\mathrm{s}\,((1 + 0.0000000035) - 1) = 0.11\ \mathrm{s} \tag{2.15}$$

2.2 The Lorentz transformations

Esercizio 3 *We have two inertial frames of reference O and O'. The coordinate axes and origins of the two systems are superimposed at the instant $t = 0\,\text{s}$. The second system moves relative to the first with speed $v = 7.3 \times 10^7\,\text{m/s}$ in the positive direction of the x axis.*

Find at which point in space-time it is located in the system O' an event that in system O it happens at the instant $t = 3\,\text{min}$ at the point $\vec{P} \equiv (45\,\text{km}; 3\,\text{km}; 2\,\text{km})$.

For the particular mutual orientation of the two reference systems we can use the simplest form of the Lorentz transformations:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \tag{2.16}$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{2.17}$$

These transformations allow us to find the coordinates (t, x, y, z) of an event in the reference system O knowing the coordinates (t', x', y', z') of the same event in the reference system O'. We replace the data:

$$\gamma = \frac{1}{\sqrt{1 - \left(\dfrac{7.3 \times 10^7\,\text{m/s}}{299792458\,\text{m/s}}\right)^2}} = 1.031 \tag{2.18}$$

$$\begin{aligned} x' &= 1.031 \cdot (45 \times 10^3\,\text{m} - 7.3 \times 10^7\,\text{m/s} \cdot 180\,\text{s}) = -1.35 \times 10^{10}\,\text{m} \\ y' &= 3 \times 10^3\,\text{m} \\ z' &= 2 \times 10^3\,\text{m} \\ t' &= 1.031 \cdot \left(180\,\text{s} - \frac{7.3 \times 10^7\,\text{m/s} \cdot 45 \times 10^3\,\text{m}}{(299792458\,\text{m/s})^2}\right) = 186\,\text{s} \end{aligned} \tag{2.19}$$

So, in the reference system O', the event happens at the instant $t' = 186\,\text{s}$, at the point of spatial coordinates $\vec{P}' \equiv (-1.35 \times 10^{10}\,\text{m}; 3 \times 10^3\,\text{m}; 2 \times 10^3\,\text{m})$.

Esercizio 4 *We have two observers, O and O′: the first stays on Earth and the second moves on a spaceship to a space station that is far away 2 light-years from Earth, proceeding at speed $v = 0.50\,\mathrm{c}$. When the spaceship sets off on its journey, the two observers are on Earth at the same time.*

1. *Find the points in space-time, for the two observers, where the journey begins.*
2. *Find the points in space-time, for the two observers, where the journey ends.*
3. *Find the duration of the journey that each of the two observers measures in their reference system, indicating which of the two durations can be interpreted as a proper time.*
4. *Find the length of the journey that each of the two observers measures in its own frame of reference, indicating which of the two can be interpreted as proper length.*

The text tells us that in the initial instant time is the same for the two observers: let us set it, at our discretion and for simplicity's sake, equal to zero.

$$t_1 = t'_1 = 0\,\mathrm{s} \tag{2.20}$$

Suppose the motion takes place in a straight line, at a constant speed. If this hypothesis holds we can consider the reference system of the Earth and of the spaceship as inertial and we can apply the Lorentz transformations. If the motion were different, we would almost certainly have to use a much more sophisticated model, perhaps general relativity.

As regards the spatial orientation, we set the axis x in the direction of the trajectory of motion and with the positive direction it agrees with the direction of the spaceship's speed. The Aces y and z are consequently perpendicular to the motion and for them relativistic phenomena are not observed: we exclude them from the discussion.

Now we can fix the initial position of the journey in the two reference systems in any coordinate of the axis x: let us set it, at our discretion and for simplicity, equal to zero for both systems. As for time they are defined up to an arbitrary constant.

$$x_1 = x'_1 = 0\,\mathrm{m} \tag{2.21}$$

1. Now we can say that the initial position in the two inertial reference systems is:

$$\vec{P}_1 = \vec{P}'_1 \equiv (0\,\mathrm{m}; 0\,\mathrm{s}) \tag{2.22}$$

2. As far as constructed and known so far, between our reference systems we can apply the following Lorentz transformations:

$$\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{\mathrm{c}^2}\right) \end{aligned} \tag{2.23}$$

As for the reference system O, the arrival point is two light years from the starting point, therefore:

$$x_2 = 2\,\mathrm{ly} = 1.8908 \times 10^{16}\,\mathrm{m} \tag{2.24}$$

The elapsed time can be derived from the speed definition.

$$v = \frac{\Delta x_0}{\Delta t} \tag{2.25}$$

where v is the speed of the spacecraft, Δx_0 the distance between the Earth and the space station, measured while standing still on the Earth, and Δt the elapsed time.

$$t_2 = t_2 - t_1 = \Delta t = \frac{\Delta x_0}{v} = \frac{2\,\text{ly}}{0.5\,\text{c}} = 4.0\,\text{years} = 1.2614 \times 10^8\,\text{s} \tag{2.26}$$

So:

$$\vec{P}_2 \equiv (x_2; t_2) = (2\,\text{ly}; 4\,\text{y}) = (1.8908 \times 10^{16}\,\text{m}; 1.2614 \times 10^8\,\text{s}) \tag{2.27}$$

For the reference system O' we substitute what we know into the transformations:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\left(\frac{0.5\,\text{c}}{\text{c}}\right)^2}} = 1.1547 \tag{2.28}$$

$$\begin{aligned} x'_2 &= 1.1547 \cdot (2\,\text{ly} - 0.5\,\text{c} \cdot 4\,\text{y}) = 0\,\text{ly} = 0\,\text{m} \\ t'_2 &= 1.1547 \cdot \left(4\,\text{y} - \frac{0.5\,\text{c} \cdot 2\,\text{ly}}{\text{c}^2}\right) = 3.45\,\text{a} = 1.0925 \times 10^8\,\text{s} \end{aligned} \tag{2.29}$$

The arrival point in the reference system O' is:

$$\vec{P}'_2 \equiv (x'_2; t'_2) = (0\,\text{m}; 1.0925 \times 10^8\,\text{s}) \tag{2.30}$$

3. For the reference system O', as we have already calculated, the journey lasts:

$$\Delta t = t_2 - t_1 = t_2 = 1.2614 \times 10^8\,\text{s} \tag{2.31}$$

For the reference system O' the journey lasts:

$$\Delta t' = t'_2 - t'_1 = t'_2 = 1.0925 \times 10^8\,\text{s} \tag{2.32}$$

The travel time for the reference system O' it's a proper time. The relationship between the two durations must also exist:

$$\Delta t' = \Delta t_0 = \frac{\Delta t}{\gamma} \tag{2.33}$$

Indeed:

$$\frac{\Delta t}{\gamma} = \frac{1.2614 \times 10^8\,\text{s}}{1.1547} = 1.0925 \times 10^8\,\text{s} = \Delta t' \tag{2.34}$$

4. For the reference system O the length of the trip L, as we know from the text, it is 2 ly. The arrival and departure positions can be measured at the same instant by the same reference system: we have a proper length.
For the reference system O' the length L' *it is not* $\Delta x' = x'_2 - x'_1$. In fact, besides being nothing, that length is between two events that for O' they don't happen at the same time. Then the length is, for example, the distance between the starting point and the finish line both seen at the time of departure.
If the motion is uniform in a straight line, the finish line is at a distance which is the product of the speed of the journey by the time it lasts.

$$L' = v \cdot t'_2 = 0.5\,\text{c} \cdot 1.0925 \times 10^8\,\text{s} = 1.64 \times 10^{16}\,\text{m} \tag{2.35}$$

Warnings

In a previous version of this file, when calculating the position x_2' have not played exact calculations, as written above, but numerical calculations, moreover with more significant figures than it would be appropriate to write.

$$x_2{}' = 1.1547 \cdot (1.8908 \times 10^{16}\,\mathrm{m} - 1.4989 \times 10^{8}\,\mathrm{m/s} \cdot 1.2614 \times 10^{8}\,\mathrm{s}) = 1.0108 \times 10^{12}\,\mathrm{m} \quad (2.36)$$

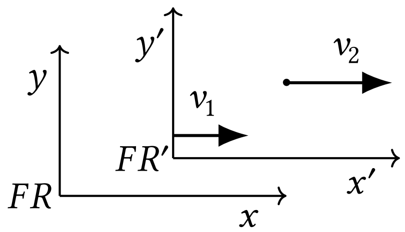
What is wrong with a result so different from the correct one? The problem is that the result is not formally incorrect. With five significant figures we can expect the result to be an approximation of $0,01\%$ and indeed it is. The ratio of the obtained distance to the length L it is of $0,005\%$, that is, it is a zero within the precision limits of the calculations. This result is also very sensitive to the number of digits used to perform the calculation: with some calculators, preserving all the mathematical precision in the previous calculations, I got as a result 2.7×10^4 m. However we don't see any particular meaning in those numbers, but we find all the physics that we have to expect only in the null result. Therefore I invite you in relativity to pay even more attention than usual in carrying out approximate calculations instead of exact ones, if these are possible, and to the number of digits used also in intermediate steps.

2.3 The composition of velocities

Esercizio 5 *The spaceship Arcadia moves away with speed* $v_1 = 0.60\,\mathrm{c}$ *compared to the Earth. Then launch a missile with a speed* $v_2 = 0.90\,\mathrm{c}$ *with respect to it, in front of it.*

Find the speed of the rocket relative to Earth.

In the absence of specific indications, let us suppose that the movement of the missile and of the spaceship all take place on the same straight line. At our discretion and as a purely formal choice, let's assume that the movement takes place on the axis x: another choice could in this case change the calculations, but not the physics. If these conditions are valid, we call FR the frame of reference of the Earth and we call FR' the ship's frame of reference. The spaceship and its frame of reference have the same speed and move in the positive direction of the axis x. The two frame of reference have parallel axes. We can represent this with this figure:



The classical composition of the velocities provides that in this case the relative velocities add up directly. Not so in relativity. If we call v_1 the speed of the second frame of reference with respect to the first (the speed of the spaceship) e v_2 the speed of an object with respect to this last reference (the speed of the missile) then this object has a speed v_E with respect to the first frame of reference (the Earth), according to this relationship:

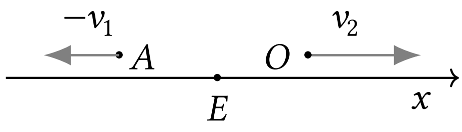
$$v_E = \frac{v_1 + v_2}{1 + \frac{v_1 \cdot v_2}{\mathrm{c}^2}} = \frac{0.6\,\mathrm{c} + 0.8\,\mathrm{c}}{1 + \frac{0.6\,\mathrm{c} \cdot 0.8\,\mathrm{c}}{\mathrm{c}^2}} = 0.95\,\mathrm{c} \tag{2.37}$$

We observe that in the previous relation the speeds are taken with a positive sign if you agree with the positive direction of the axis (as in this case) or with a negative sign otherwise.

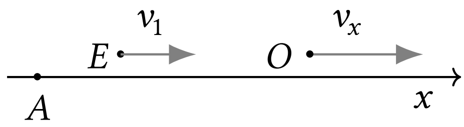
Esercizio 6 *The Arcadia spaceship and the Orion move away in opposite directions from the Beta space base. The space station reports the first spacecraft moving away at speed* $v_1 = 0.60\,\mathrm{c}$ *and the second with speed* $v_2 = 0.40\,\mathrm{c}$.

How fast is Orion moving away from Arcadia?

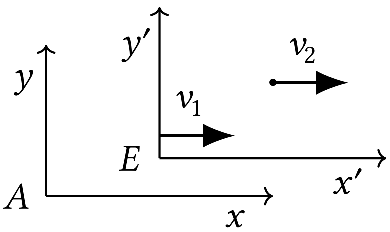
The text tells us that the motion of spaceships occurs in the same direction: we represent the motion all on the axis x. The speed of the Arcadia is indicated with the minus in front because it is in the opposite direction to the positive direction of the axis x.



From the point of view of Arcadia, the Earth is moving away from it with speed v_1 (the reciprocal velocities of two objects are always the same), but in opposite directions. The Orion departs Arcadia with unknown speed v_1.



This representation fully corresponds to the one described in the previous exercise, where the frame reference FR is that of Arcadia and FR' it is that of the earth.



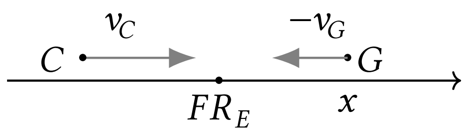
Concluding:

$$v_x = \frac{v_1 + v_2}{1 + \frac{v_1 \cdot v_2}{c^2}} = \frac{0.6\,c + 0.9\,c}{1 + \frac{0.6\,c \cdot 0.9\,c}{c^2}} = 0.97\,c \tag{2.38}$$

Esercizio 7 *The spaceship Cavour, if seen from the Earth, travels* 18×10^7 km *in* 8.2×10^2 s *moving away from it. The spaceship Garibaldi is traveling towards the Earth at speed* $v_G = 0.60\,c$ *in the same direction.*

1. *Find the speed of Cavour as seen from Earth.*
2. *Find the speed of the Cavour as seen from the Garibaldi.*
3. *Find the distance traveled and the time taken by the Cavour when viewed from the Garibaldi.*

We call FR_E the reference frame of the Earth, FR_C that of Cavour e FR_G that of Garibaldi. The motion all happens in one dimension. We represent what is illustrated by the text by imagining Cavour moving in the positive direction of the axis x and the Garibaldi in the negative direction when viewed from FR_E.

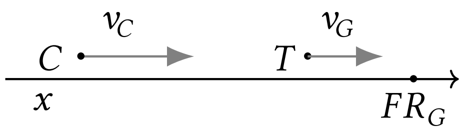


1. The speed of the Cavour seen from the Earth is simply the ratio between the distance covered and the time taken: both this distance and this time are both measured by the Earth's reference system.

$$v_C = \frac{x}{t} = \frac{\Delta x_0}{\Delta t} = \frac{18 \times 10^{10}\ \mathrm{m}}{8.2 \times 10^2\ \mathrm{s}} = 2.20 \times 10^8\ \mathrm{m/s} = 0.732\,c \tag{2.39}$$

The distance seen from the Earth can be considered a proper length: we can put a fixed ruler that goes from the starting position of the Cavour to the arrival point. Elapsed time is not proper time: it refers to two events that do not take place in the same place.

2. From the text we can see that Cavour and Garibaldi move towards each other in the same direction. Their relative velocity is related to the sum of their velocities relative to the Earth. We represent what you see from FR_G.



So the speed v_{CG} of the Cavour seen from Garibaldi is:

$$v_{CG} = \frac{v_C + v_G}{1 + \dfrac{v_C \cdot v_G}{c^2}} = \frac{0.732\,c + 0.6\,c}{1 + \dfrac{0.732\,c \cdot 0.6\,c}{c^2}} = 0.926\,c \tag{2.40}$$

3. To answer the third question we use two distinct methods.

First method
We cannot directly apply the formulas relating to length contraction and time dilation because these can only be applied if in one of the two frame the measured length or time interval is proper.
In particular, the time measured by the Earth and that measured by Garibaldi are not both: we cannot establish a direct relationship between the two time intervals. The time measured on the Cavour is instead a proper time. We write a relationship between the time measured on Earth and that measured on Cavour obtaining the latter.

$$\gamma_1 = \frac{1}{\sqrt{1 - \left(\dfrac{v_C}{c}\right)^2}} = \frac{1}{\sqrt{1 - \left(\dfrac{0.732\,c}{c}\right)^2}} = 1.47 \tag{2.41}$$

$$\begin{aligned} \Delta t_E &= \gamma_1 \Delta t_{0\,C} \\ \Delta t_{0\,C} &= \frac{\Delta t_E}{\gamma_1} = \frac{8.2 \times 10^2\,\text{s}}{1.47} = 5.58 \times 10^2\,\text{s} \end{aligned} \tag{2.42}$$

The time interval measured on the Garibaldi, which is not proper time, can in turn be related to the time measured on the Cavour, using their relative velocity that we have previously derived.

$$\gamma_2 = \frac{1}{\sqrt{1 - \left(\dfrac{v_{CG}}{c}\right)^2}} = \frac{1}{\sqrt{1 - \left(\dfrac{0.926\,c}{c}\right)^2}} = 2.65 \tag{2.43}$$

$$\Delta t_G = \gamma_2 \Delta t_{0\,C} = 2.65 \cdot 5.58 \times 10^2\,\text{s} = 1.48 \times 10^3\,\text{s} \tag{2.44}$$

To derive the length measured by the Garibaldi we can apply the definition of speed.

$$\begin{aligned} v_{CG} &= \frac{\Delta x_G}{\Delta t_G} \\ \Delta x_G &= v_{CG} \cdot \Delta t_G = 0.926\,c \cdot 1.48 \times 10^3\,\text{s} = 4.11 \times 10^{11}\,\text{m} \end{aligned} \tag{2.45}$$

We observe that to derive this length textitwe could not have used the usual formulas related to the contraction of lengths. In fact, in our case the length measured by the Earth can be considered a proper length as the starting and arrival position can be measured at the same time: we can therefore find the contracted length measured by Cavour. Once we know this, however, we can not make any transformation of the same type to know what distance the Garibaldi measures. We cannot even make a direct transformation between the distance measured by the Earth and that measured by Garibaldi.

Secondo metodo

We can obtain the same results more directly by using Lorentz transformations. In order to use them, we assume at our discretion, without modifying the physics of the problem, that the three reference frames have the origin of the coincident axes at the instant $t = 0$ s, where t is specific for each frame. In particular, we define the superscript coordinates measured in the FR_E and the superscript coordinates those measured in the FR_G.
The initial position is therefore, for all, at the coordinate $x_1 = x_1' = 0$ m at time $t_1 = t_1' = 0$ s.
The distance measured by FR_E is:

$$\Delta x = x_2 - x_1 = 18 \times 10^{10}\,\mathrm{m} - 0\,\mathrm{m} = 18 \times 10^{10}\,\mathrm{m} \tag{2.46}$$

Considering that:

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v_G}{\mathrm{c}}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{0.6\,\mathrm{c}}{\mathrm{c}}\right)^2}} = 1.25 \tag{2.47}$$

Then the final position measured from FR_G is:

$$x_2' = \gamma(x_2 - v_G\,t_2) = 1.25 \cdot (18 \times 10^{10}\,\mathrm{m} - (-0.6\,\mathrm{c}) \cdot 8.2 \times 10^2\,\mathrm{s}) = 4.09 \times 10^{11}\,\mathrm{m} \tag{2.48}$$

The velocity of Garibaldi is negative because from Earth it moves to the left in the negative direction of the axis x. The value is different from that calculated with the first method due to the greater rounding performed in the previous steps.
The distance measured from FR_G is:

$$\boldsymbol{\Delta x_G} = \Delta x' = x_2' - x_1' = 4.09 \times 10^{11}\,\mathrm{m} - 0\,\mathrm{m} = 4.09 \times 10^{11}\,\mathrm{m} \tag{2.49}$$

Similarly the elapsed time measured by FR_T is:

$$\Delta t = t_2 - t_1 = 8.2 \times 10^2\,\mathrm{s} - 0\,\mathrm{s} = 8.2 \times 10^2\,\mathrm{s} \tag{2.50}$$

The final instant measured from FR_G is:

$$t_2' = \gamma\left(t_2 - \frac{v_G \cdot x_2}{\mathrm{c}^2}\right) = 1.25 \cdot \left(8.2 \times 10^2\,\mathrm{s} - \frac{(-0.6\,\mathrm{c}) \cdot 18 \times 10^{10}\,\mathrm{m}}{\mathrm{c}^2}\right) = 1.48 \times 10^3\,\mathrm{s} \tag{2.51}$$

The time interval measured by FR_G is:

$$\boldsymbol{\Delta t_G} = \Delta t' = t_2' - t_1' = 1.48 \times 10^3\,\mathrm{s} - 0\,\mathrm{s} = 1.48 \times 10^3\,\mathrm{s} \tag{2.52}$$

2.4 Momentum

Esercizio 8 *Two particles, mass* $m_1 = 2.37 \times 10^{-30}$ kg *and* $m_2 = 8.26 \times 10^{-31}$ kg, *move towards each other, towards the same observer, in the same direction, respectively with speed* $v_1 = 0.80$ c *and* $v_2 = 0.60$ c. *The particles collide in a completely inelastic way.*

Find the final velocity of the two particles after the collision.

Also in relativity the total momentum of a system is conserved if the system is not subjected to external forces. If we consider that the two bodies remain united with the same speed after the collision we can write:

$$p_1 + p_2 = p_f \tag{2.53}$$

We calculate the initial momenta of the two particles.

$$p_1 = \gamma m_1 v_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} mv = \frac{1}{\sqrt{1-(0.8)^2}} \cdot 2.37 \times 10^{-30}\,\text{kg} \cdot 0.8\,\text{c} = 9.47 \times 10^{-22}\,\text{kg m s}^{-1} \tag{2.54}$$

$$p_2 = \gamma m_2 v_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} mv = \frac{1}{\sqrt{1-(0.6)^2}} \cdot 8.26 \times 10^{-31}\,\text{kg} \cdot 0.6\,\text{c} = 1.86 \times 10^{-22}\,\text{kg m s}^{-1} \tag{2.55}$$

We observe that the particles move in opposite directions: one of the momenta (at our discretion) has a negative sign.

$$p_f = \gamma m_{tot} v_f = p_1 + (-p_2) = 7.62 \times 10^{-22}\,\text{kg m s}^{-1} \tag{2.56}$$

$$m_{tot} = 2.37 \times 10^{-30}\,\text{kg} + 8.26 \times 10^{-31}\,\text{kg} = 3.20 \times 10^{-30}\,\text{kg} \tag{2.57}$$

We obtain v_f from this equation, where it is the only unknown.

$$\begin{gathered} \frac{p_f}{m_{tot}} = \gamma v_f = \frac{v_f}{\sqrt{1-\frac{v_f^2}{c^2}}} \\ \frac{p_f^2}{m_{tot}^2} = \frac{v_f^2}{1-\frac{v_f^2}{c^2}} = \frac{v_f^2}{\frac{c^2-v_f^2}{c^2}} = \frac{v_f^2 c^2}{c^2-v_f^2} \end{gathered} \tag{2.58}$$

$$\begin{gathered} \frac{p_f^2}{m_{tot}^2} - \frac{v_f^2 c^2}{c^2-v_f^2} = 0 \\ \frac{p_f^2(c^2-v_f^2) - v_f^2 c^2 m_{tot}^2}{(c^2-v_f^2)m_{tot}^2} = 0 \\ p_f^2 c^2 - p_f^2 v_f^2 - m_{tot}^2 v_f^2 c^2 = 0 \\ p_f^2 c^2 = v_f^2(p_f^2 + m_{tot}^2 c^2) \\ v_f = \sqrt{\frac{p_f^2 c^2}{p_f^2 + m_{tot}^2 c^2}} \end{gathered} \tag{2.59}$$

Finally:

$$v_f = \sqrt{\frac{(7.62 \times 10^{-22}\,\mathrm{kg\,m\,s^{-1}})^2 \cdot v^2}{(7.62 \times 10^{-22}\,\mathrm{kg\,m\,s^{-1}})^2 + \mathrm{c}^2 \cdot (3.20 \times 10^{-30}\,\mathrm{kg})^2}} = 1.86 \times 10^8\,\mathrm{m/s} = 0.62\,\mathrm{c} \quad (2.60)$$

2.5 Energy

Esercizio 9 *Find the rest energy, kinetic energy, and total energy of a muon moving at speed* $v = 0.70\,\mathrm{c}$. *[mass muon* $m_\mu = 105.7\,\mathrm{MeV/c^2}$*]*

In nuclear and particle physics it is very common to use these alternative units of measure for energy, momentum and mass. In particular, the mass is often indicated in $\mathrm{MeV/c^2}$ where $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$.

The rest energy E_0 of a mass in relativity it is:

$$E_0 = m\mathrm{c}^2 \quad (2.61)$$

whereby:

$$m = \frac{E_0}{\mathrm{c}^2} \quad (2.62)$$

which has the same form as the unit used here for the mass of the muon. In our case:

$$E_0 = (105.7\,\mathrm{MeV/c^2}) \cdot \mathrm{c}^2 = 105.7\,\mathrm{MeV} = 105.7 \times 10^6 \cdot 1.602 \times 10^{-19}\,\mathrm{J} = 1.693 \times 10^{-11}\,\mathrm{J} \quad (2.63)$$

The kinetic energy E_c is defined as:

$$E_c = (\gamma - 1)m\mathrm{c}^2 = \left(\frac{1}{\sqrt{1 - \frac{v^2}{\mathrm{c}^2}}} - 1\right) E_0 = \left(\frac{1}{\sqrt{1 - (0.7)^2}} - 1\right) \cdot 1.693 \times 10^{-11}\,\mathrm{J} = 6.778 \times 10^{-12}\,\mathrm{J} \quad (2.64)$$

The total energy is the sum of the rest energy and the kinetic energy:

$$E_{tot} = E_0 + E_c = \gamma m\mathrm{c}^2 = 2.371 \times 10^{-11}\,\mathrm{J} \quad (2.65)$$

Esercizio 10 *A particle of mass* $m = 205\,\mathrm{MeV/c^2}$ *is sent to a particle detector.*
The detector tells us that the momentum of the particle is $p = 365\,\mathrm{MeV/c}$.
Determine the total energy and velocity of the particle, in the given units and in the SI.

The relativistic relationship between total energy, mass, and momentum is this:

$$E^2 = m^2 c^4 + p^2 c^2 \tag{2.66}$$

In this case:

$$\begin{gathered} E = \sqrt{\left(205\,\frac{\mathrm{MeV}}{\mathrm{c}^2}\right)^2 \mathrm{c}^4 + \left(365\,\frac{\mathrm{MeV}}{\mathrm{c}}\right)^2 \mathrm{c}^2} = \\ \sqrt{(205\,\mathrm{MeV})^2 + (365\,\mathrm{MeV})^2} = 419\,\mathrm{MeV} = 6.71 \times 10^{-11}\,\mathrm{J} \end{gathered} \tag{2.67}$$

From the relation:

$$p = mv\gamma \tag{2.68}$$

we proceed as in exercise 8 and write directly:

$$v_f = \sqrt{\frac{p_f^2 c^2}{p_f^2 + m_{tot}^2 c^2}} \tag{2.69}$$

$$v_f = \sqrt{\frac{\left(365\,\frac{\mathrm{MeV}}{\mathrm{c}}\right)^2 \mathrm{c}^2}{\left(365\,\frac{\mathrm{MeV}}{\mathrm{c}}\right)^2 + \left(205\,\frac{\mathrm{MeV}}{\mathrm{c}^2}\right)^2 \mathrm{c}^2}} = \sqrt{\frac{(365\,\mathrm{MeV})^2}{\left(365\,\frac{\mathrm{MeV}}{\mathrm{c}}\right)^2 + \left(205\,\frac{\mathrm{MeV}}{\mathrm{c}}\right)^2}} = 0.87\,\mathrm{c} = 2.61 \times 10^8\,\mathrm{m/s} \tag{2.70}$$

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